International Journal of Theoretical Physics, Vol. 26, No. 6, 1987

Spinor Fields in the Nonsymmetric, Non-Abelian Kaluza–Klein Theory

M. W. Kalinowski[1]

Received February 8, 1987

Spinor fields are considered in the framework of the nonsymmetric Kaluza-Klein theory and the nonsymmetric Jordan-Thiry theory (in non-Abelian case). Dipole moments for fermions of value 10^{-31} and pseudomass-like terms are found.

1. INTRODUCTION

We deal with spinor fields in the framework of the nonsymmetric Kaluza-Klein theory and the nonsymmetric Jordan-Thiry theory. To do this we introduce on **P**, an $(n+4)$-dimensional Kaluza-Klein manifold [in the nonsymmetric case, see Kalinowski (1983a–c, 1984a)], a spinor field belonging to the fundamental representation of a group $SO(1, n+3)$.

Using a minimal coupling scheme from Moffat's theory of gravitation (see Kalinowski, 1986), we introduce for this spinor field a new kind of gauge derivative as in Kalinowski (1981a, b, 1982, 1983d, 1984b, 1987). Simultaneously, we use the dimensional reduction procedure from Kalinowski (1982, 1984b) for this spinor field.

In the Lagrangian for this field we get new terms, which we interpret as the interaction of dipole moments with the Yang-Mills field and pseudomass-like terms. In the case $\dim G = n = 2l+1$, we can interpret some of these terms as the interaction of the dipole electric moment of the fermion with the electromagnetic field. Thus we get *PC*-breaking, as in Thirring (1972), Kalinowski (1981a, 1984b, 1987).

This paper is organized as follows. In Section 2 we describe some elements of the nonsymmetric Kaluza-Klein theory and the nonsymmetric Jordan-Thiry theory. In Section 3 we introduce a dimensional reduction

[1]Department of Physics, University of Toronto, Toronto, Ontaria, Canada M5S 1A7; and Institute of Theoretical Physics, University of Warsaw, 00-681 Warsaw, Poland.

0020-7748/87/0600-0559$05.00/0 © 1987 Plenum Publishing Corporation

procedure. In Section 4 we describe minimal coupling between Dirac's field and geometry (gravity) in the Moffat theory of gravitation. In Section 5 we introduce a new gauge derivative for a spinor field Ψ and generalize minimal coupling scheme. We get new terms in the lagrangian. In the Appendix we deal with elements of Clifford algebras that we use in the paper.

2. THE NONSYMMETRIC KALUZA–KLEIN THEORY AND THE NONSYMMETRIC JORDAN–THIRY THEORY

Let **P** be the principal fiber bundle with the structural group G, over space-time E with a projection π, and let us define on this bundle a connection ω. Let us suppose that G is semisimple and that its Lie algebra $\mathfrak{g}$ has a real representation such that $\mathrm{Tr}[(X_a)^2]$ is not equal to zero for every a. Here Tr is understood in the sense of the representation space of the Lie algebra $\mathfrak{g}$, and X_a are generators of $\mathfrak{g}$. On space-time E we define a nonsymmetric metric tensor such that

$$g_{\alpha\beta} = g_{(\alpha\beta)} + g_{[\alpha\beta]} \tag{1}$$

$$g_{\alpha\beta} g^{\gamma\beta} = g_{\beta\alpha} g^{\beta\gamma} = \delta^{\gamma}_{\alpha} \tag{2}$$

where the order of indices is important. We define on E two connections $\bar{\omega}^{\alpha}_{\beta}$ and $\bar{W}^{\alpha}_{\beta}$:

$$\bar{\omega}^{\alpha}_{\beta} = \bar{\Gamma}^{\alpha}_{\beta\gamma}\bar{\theta}^{\gamma}, \qquad \bar{W}^{\alpha}_{\beta} = \bar{W}^{\alpha}_{\beta\gamma}\bar{\theta}^{\gamma} \tag{3}$$

$$\bar{W}^{\alpha}_{\beta} = \bar{\omega}^{\alpha}_{\beta} - \tfrac{2}{3}\delta^{\alpha}_{\beta}\bar{W} \tag{4}$$

where

$$\bar{W} = \bar{W}_{\gamma}\bar{\theta}^{\gamma} = 1/2(\bar{W}^{\sigma}_{.\gamma\sigma} - \bar{W}^{\sigma}_{.\sigma\gamma})\bar{\theta}^{\gamma}$$

For the connection $\bar{\omega}^{\alpha}_{\beta}$ we suppose the following condition:

$$\bar{D}g_{\alpha+\beta-} = \bar{D}g_{\alpha\beta} - g_{\alpha\delta}\bar{Q}^{\delta}_{\beta\gamma}(\bar{\Gamma}) = 0, \qquad \bar{Q}^{\alpha}_{\beta\alpha}(\bar{\Gamma}) = 0 \tag{5}$$

where $\bar{D}$ is the exterior covariant derivative with respect to $\bar{\omega}^{\alpha}_{\beta}$ and $\bar{Q}^{\alpha}_{\beta\gamma}(\bar{\Gamma})$ is the torsion of $\bar{\omega}^{\alpha}_{\beta}$. Thus, we have on space-time E all quantities from Moffat's theory of gravitation (Moffat, 1979, 1981, 1982). The exterior covariant derivative with respect to $\bar{W}^{\lambda}_{\mu}$ we will denote D_w. Now we define on P the natural frame

$$\theta^{A} = (\pi^*(\bar{\theta}^{\alpha}), \theta^{\alpha} = \lambda\omega^{a}), \qquad \lambda = \mathrm{const} \tag{6}$$

where $\omega = \omega^{a}X_a$ is a connection on **P**. The two-form of curvature of connection ω is

$$\Omega = \mathrm{hor}\, d\omega = 1/2 H^{a}_{\mu\nu}\theta^{\mu}_{\Lambda}\theta^{\nu}X_a \tag{7}$$

Ω obeys the structural Cartan equation

$$\Omega = d\omega + 1/2[\omega, \omega] \tag{8}$$

Bianchi's identity for ω is

$$\text{hor } d\Omega = 0 \tag{9}$$

Horizontality is understood in the sense of the connection ω on **P**. The map $e: E \supset U \to P$, so that $e \circ \pi = id$ is called a cross section. From the physical point of view it means choosing a gauge. Thus,

$$e^*\omega = e^*(\omega^a X_a) = A^a_\mu \bar{\theta}^\mu X_a \tag{10}$$

$$e^*\Omega = e^*(\Omega^a X_a) = 1/2 F^a_{\mu\nu} \bar{\theta}^\mu_\wedge \bar{\theta}^\nu X_a \tag{11}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + C^a_{bc} A^b_\mu A^c_\nu \tag{12}$$

X_a, $a = 1, 2, \ldots, n = \dim G$, are generators of the Lie algebra of the group G and

$$[X_a, X_b] = C^c_{ab} X_c \tag{13}$$

A covariant derivation on **P** with respect to ω, d^{guage} is defined as follows:

$$\overset{\text{guage}}{d}\ \Psi = \text{hor } d\Psi \tag{14}$$

This derivation is called "gauge" derivation, where Ψ is, for example, a spinor field on **P**.

It is convenient to introduce the following notations. Capital Latin indices A, B, C run over $1, 2, 3, 4, \ldots, n+4$, $n = \dim G$; lowercase Greek indices $\alpha, \beta, \gamma = 1, 2, 3, 4$; and lowercase Latin indices $a, b, c, d = 5, 6, \ldots, n+4$. A bar over θ^α and ω^α_β indicates that both quantities are defined on E. According to Kalinowski (1983a, b), we introduce on **P** the natural nonsymmetric tensor

$$\gamma_{AB} = \left(\begin{array}{c|c} g_{\alpha\beta} & 0 \\ \hline 0 & l_{ab} \end{array}\right) \tag{15}$$

where

$$l_{ab} = h_{ab} + \mu K_{ab} \tag{16}$$

with $h_{ab} = C^c_{ad} C^d_{cb}$ a Killing tensor on G, and $K_{ab} = C^c_{ab} \operatorname{Tr}[(X_c)^2]$; μ is a dimensionless constant [see Kalinowski (1983a, b) for more details]. In the case of the nonsymmetric Jordan-Thiry theory we define on **P** the following object:

$$\gamma_{AB} = \left(\begin{array}{c|c} g_{\alpha\beta} & 0 \\ \hline 0 & \rho^2 l_{ab} \end{array}\right) \tag{17}$$

where ρ is the scalar field on E [see Kalinowski (1983c, 1984a) for more details].

We suppose that $\det(l_{ab}) \neq 0$. Now we define on **P** a connection ω^A_B such that

$$D\gamma_{A+B-} = D\gamma_{AB} - \gamma_{AD} Q^D_{BC}(\Gamma)\theta^C = 0 \tag{18}$$

where

$$\omega^A_B = \Gamma^A_{BC}\theta^C$$

D is the exterior covariant derivative with respect to the connection ω^A_B and $Q^D_{BC}(\Gamma)$ is the tensor of torsion for the connection ω^A_B. In the case of the nonsymmetric Kaluza-Klein theory we get

$$\omega^A_B = \left(\begin{array}{c|c} \pi^*(\bar{\omega}^\alpha_\beta) - \frac{1}{2}\lambda l_{db} g^{\mu\alpha} L^d_{\mu\beta}\theta^b & \frac{1}{2}\lambda L^a_{\beta\gamma}\theta^\gamma \\ \hline \frac{1}{2}\lambda l_{bd} g^\alpha_\beta (2H^d_{\gamma\beta} - L^d_{\gamma\beta})\theta^\gamma & (2/\lambda)\tilde{\omega}^a_b \end{array}\right) \tag{19}$$

where

$$L^d_{\gamma\beta} = -L^d_{\beta\gamma} \tag{20}$$

is a tensor on P such that

$$l_{dc} g_{\mu\beta} g^{\gamma\mu} L^d_{\gamma\alpha} + l_{cd} g_{\alpha\mu} g^{\mu\gamma} L^d_{\beta\gamma} = 2 l_{cd} g_{\alpha\mu} g^{\mu\gamma} H^d_{\beta\gamma} \tag{21}$$

$$\tilde{\omega}^a_b = \tilde{\Gamma}^a_{bc}\theta^c \tag{22}$$

and

$$l_{db}\tilde{\Gamma}^d_{ac} + l_{ad}\tilde{\Gamma}^d_{cb} = 1/2 l_{ad} C^d_{bc} \tag{23}$$

$$\tilde{\Gamma}^d_{ac} = -\tilde{\Gamma}^d_{ca} \tag{24}$$

$$\tilde{\Gamma}^a_{ba} = 0 \tag{25}$$

[see Kalinowski (1983a, b) for more details]. In the case of the nonsymmetric Jordan-Thiry theory we get

$$\omega^A_B = \left[\begin{array}{c|c} \pi^*(\bar{\omega}^\alpha_\beta) - \frac{1}{2}\lambda\rho^2 l_{db} g^{\delta\alpha} L^d_{\delta\beta}\theta^b & \frac{1}{2}\lambda L_{\beta\gamma}\theta^\gamma + (1/\rho) g_{\beta\delta}\tilde{g}^{(\delta\gamma)}\rho_{,\gamma}\theta^a \\ \hline \frac{1}{2}\lambda\rho^2 l_{bd} g^{\alpha\beta}(2H^d_{\gamma\beta} - L^d_{\gamma\beta})\theta^\gamma & (1/\rho) g_{\delta\beta}\tilde{g}^{(\delta\gamma)}\rho_{,\gamma}\delta^a_b\theta^\beta_{+(2/\lambda)}\tilde{\omega}^a_b \\ -\rho\tilde{g}^{(\alpha\beta)}\rho_{,\beta} l_{bc}\theta^c & \end{array}\right] \tag{26}$$

where $\tilde{g}^{(\alpha\beta)}$ is the inverse tensor for $g_{(\alpha\beta)}$,

$$g_{(\alpha\beta)}\tilde{g}^{(\alpha\gamma)} = \delta^\gamma_\beta \tag{27}$$

[For more details see Kalinowski (1983c, 1984a).] In the Kaluza-Klein theory and in the Jordan-Thiry theory $\lambda = 2G^{1/2}/c^2$, where G is the gravitational constant and c is the velocity of light in vacuum. This condition originates from the consistency between the equation in the theory and the Einstein equation (Kaluza, 1921; Lichnerowicz, 1955a; Raysk, 1965; Kerner, 1968; Cho, 1975; Kalinowski, 1983e). In the Jordan–Thiry theory there exists the effective gravitational constant

$$G_{\text{eff}} = G\rho^3 \tag{28}$$

and ρ plays the role of the gravitational "constant," which now depends on a point of E (Kalinowski, 1983c, 1984a). Now we define the dual Cartan base on E.

Let $\bar{\eta}_{1234} = (-\det g)^{1/2}$; $\bar{\eta}_{\alpha\beta\gamma\delta}$ is a Levi-Civita symbol and

$$\bar{\eta}_\alpha = \tfrac{1}{6}\bar{\theta}^\beta \wedge \bar{\theta}^\gamma \wedge \bar{\theta}^\delta \bar{\eta}_{\alpha\beta\gamma\delta}, \qquad \bar{\eta} = 1/4\bar{\theta}^\alpha \wedge \eta_\alpha \tag{29}$$

For details concerning elements of the geometry mentioned here see Kobayashi and Nomizu (1963), Lichnerowicz (1955b), Trautman (1970).

3. DIMENSIONAL REDUCTION

Let us consider the group $SO(1, n+3)$ and its fundamental (complex) representation of dimension $K = 4 \cdot 2^{[n/2]}$, where $[n/2] = l$ for $n = 2l$ or $2l+1$,

$$U(g)\Psi(X) = D^F(g)\Psi(g^{-1}X) \tag{30}$$
$$X \in M^{(1,n+3)}, \qquad g \in SO(1, n+3)$$

$SO(1, n+3)$ acts linearly in $M^{(1,n+3)}$ [$(n+4)$-dimensional Minkowski space]. The Lorentz group $SO(1,3) \subset SO(1, n+3)$. Thus, after restriction of g to the subgroup $SO(1,3)$ we obtain a decomposition of D^F according to (Barut and Raczka, 1977)

$$D^F|_{SO(1,3)}(\Lambda) = L(\Lambda) \underbrace{\oplus \cdots \oplus}_{[n/2]\,\text{times}} L(\Lambda), \qquad \Lambda \in SO(1,3) \tag{31}$$

where

$$L(\Lambda) = D^{(1/2,0)} \oplus D^{(0,1/2)}(\Lambda)$$

is the Dirac representation of $SO(1,3)$. The decomposition (31) for a spinor Ψ has the form

$$\Psi|_{SO(1,3)} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_2[n/2] \end{pmatrix} \tag{32}$$

where ψ_i, $i = 1, 2, \ldots, 2^{[n/2]}$, are spinors belonging to the Dirac representation ($L = D^{(1/2,0)} \oplus D^{(0,1/2)}$). Thus, due to the decomposition (32), we get a tower of 1/2-spin fermions.

More precisely, we deal with representations of Spin$(1, n+3)$ and Spin$(1,3) \approx SL(2, \mathbb{C})$.

Let us turn to the manifold P. It is a metric manifold (P, γ) with a metric tensor γ. At every point $p \in P$ we have a tangent space $T_p(P) \simeq M^{(1,n+3)}$. Let $\Psi: P \to \mathbb{C}^K$ $(K = 2^{[n/2]})$ be a spinor field on P at every point $p \in P$ belonging to fundamental representation D^F of group $SO(1, n+3)$.

For spinor field Ψ we suppose the following action of group G:

$$\Psi(pg_1) = \sigma(g_1^{-1})\Psi(p) \tag{33}$$

where $p = (x, g) \in P$; $g, g_1 \in G$. Here σ is a representation of group G in $4 \cdot 2^{[n/2]}$-dimensional complex space.

If we take a section $e: E \to P$, we get a spinor field $\Psi(e(x))$ on the manifold E (space-time). This means that at every point $x \in E$ we have, after restriction to $SO(1,3)$, the spinor $\Psi|_{SO(1,3)}$ and for it the decomposition (32) is valid. Thus

$$(e^*\Psi)|_{SO(1,3)}(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \vdots \\ \psi_{2^{[n/2]}}(x) \end{pmatrix} \tag{34}$$

Spinor fields $\Psi_i(x)$, $i = 1, 2, \ldots, 2^{[n/2]}$, are spinor fields at every point $x \in E$ belonging to the Dirac representation $L = D^{(0,1/2)} \oplus D^{(1/2,0)}$. Such a procedure we will call the dimensional reduction for a spinor field. In this way we get a tower of Dirac spinor fields on E. The following scheme symbolizes it:

$$\Psi \xrightarrow[\text{section of } P]{e} e^*\Psi \xrightarrow[\substack{\text{from } SO(1,\, n+3) \\ \text{to } SO(1,3)}]{\text{restriction}} e^*\Psi|_{SO(1,3)} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_{2^{[n/2]}} \end{pmatrix} \tag{35}$$

In Kalinowski (1981a, b, 1987) we dealt with (in a similar context) the five-dimensional (electromagnetic) case $[G = U(1), n = 1]$. Thus, we have the de Sitter group $SO(1,4)$ and we deal with the spinor Ψ belonging to the fundamental representation of group Spin$(1,4) \simeq Sp(4)$. But for this case we have $\dim D^F = \dim D^F|_{SO(1,3)}$ and after dimensional reduction we get only one spinor field on E. The procedure (35) explains the construction given in Kalinowski (1981a, b, 1987). This procedure shows how to obtain a set of Dirac spinor fields ψ_i on E if one has a spinor field on P (with a special dependence of higher group dimensions). But from the physical point of view the opposite case is more interesting. We have several spinor fields on E with which we connect physical fermion fields. From time to time it is possible to build a tower from these physical spinor fields. Some attempts have been made in constructing such towers (Rayski, 1977; Furlan and Raczka, 1980; Pati, 1980; Kim, 1987). Thus, from a physical point of view it would be interesting to describe physical fermions as a spinor field on P belonging to a fundamental representation of $SO(1, n+3)$ [Spin(1, $n+3$)]. This might be of help in understanding the generations of fermions. Now it is difficult to proceed because a group G [gauge group for GUT (Kim, 1987)] is not well established and one suspects that many new generations are possible. We know from an asymptotic freedom argument in QCD that the number of generations may be smaller than 9 (greater than 2).

3. DIRAC LAGRANGIAN IN MOFFAT'S THEORY OF GRAVITATION

In Kalinowski (1986) we found the minimal coupling scheme for the Dirac field in the Moffat theory of gravitation. We get the Lagrangian

$$L(W, \psi) = \tfrac{1}{2} i\hbar c(\bar{\psi} l \wedge \hat{D}\psi + \hat{D}\psi \wedge l\psi) + mc\eta\bar{\psi}\psi \tag{36}$$

where $l = \gamma^{\mu}\bar{\eta}_{\mu}$ and

$$\hat{D}\psi = D_w\psi - \frac{in\varepsilon_F}{3}\left(\frac{a}{l_{\rm pl}}\right)^2 \bar{W}\psi, \qquad D\bar{\psi} = D_w\bar{\psi} + \frac{in\varepsilon_F}{3}\left(\frac{a}{l_{\rm pl}}\right)^2 \bar{W}\bar{\psi} \tag{37}$$

where a is a coupling constant for fermion current in the Moffat theory (Moffat, 1979, 1981, 1982), $l_{\rm pl}$ is the Planck length $l_{\rm pl} = (G/\hbar c)^{1/2} \approx 10^{-33}$ cm, n is a nonzero integer, and $\varepsilon_F^2 = 1$; D_w is the exterior covariant derivative with respect to the connection $\bar{W}^{\lambda}_{\mu}$. In Kalinowski (1986) we proved that the Lagrangian (36) is equal to

$$L(W, \psi) = \frac{i\hbar c}{2}\left[\bar{\psi} l \wedge \left(\bar{D}\psi - \frac{in\varepsilon_F}{3}\bar{W}\psi\right) + \left(\bar{D}\bar{\psi} + \frac{in\varepsilon_F}{3}\bar{W}\bar{\psi}\right) \wedge l\psi\right]$$
$$+ mc\eta\bar{\psi}\psi \tag{38}$$

where

$$\bar{D}\psi = d\psi + \bar{\omega}^{\alpha}_{\beta}\sigma^{\beta}_{\alpha}\psi, \qquad \bar{D}\bar{\psi} - d\bar{\psi} - \bar{\psi}\sigma^{\beta}_{\alpha}\bar{\omega}^{\alpha}_{\beta} \tag{39}$$

and σ^{β}_{α} satisfies the following properties:

$$\sigma^{\beta}_{\beta} = 0 \tag{40}$$

$$2[\sigma^{\mu}_{\nu}, \sigma^{\kappa}_{\lambda}] = \delta^{\kappa}_{\nu}\sigma^{\mu}_{\lambda} - \eta_{\nu\lambda}\sigma^{\mu\kappa} + \delta^{\mu}_{\lambda}\sigma^{\kappa}_{.\nu} - \eta^{\mu\kappa}\sigma_{\nu\lambda} \tag{41}$$

$$[\sigma^{\mu}_{\nu}, \gamma^{\rho}] = 1/2(\delta^{\rho}_{\nu}\gamma^{\mu} - \eta^{\rho\mu}\gamma_{\nu}) \tag{42}$$

γ^μ are ordinary Dirac matrices satisfying a conventional relationship,

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{43}$$

$$\eta_{\nu\lambda}\eta^{\lambda\kappa} = \delta^\nu_\kappa, \qquad \gamma^\mu = \eta^{\alpha\beta}\gamma_\beta, \qquad \sigma^{\mu\kappa} = \eta^{\kappa\nu}\sigma^\mu_\nu, \qquad \sigma_{\nu\lambda} = \eta_{\nu\mu}\sigma^\mu_{.\lambda} \tag{44}$$

The contragradient spinor $\bar{\psi}$ is defined by

$$\bar{\psi} = \psi^+\beta, \qquad \beta^+ = \beta \tag{45}$$

where

$$\gamma^{\mu+} = \beta\gamma^\mu\beta^{-1} \tag{46}$$

and

$$\sigma^{\mu+}_{.\lambda} = -\beta\sigma^\mu_\lambda\beta^{-1} \tag{47}$$

The superscript plus sign denotes Hermitian conjugation. The spinor ψ was defined in Kalinowski (1986) as a 0-form of Σ-type:

$$\Sigma\colon \underset{[\text{or } GL_\downarrow(4, R)]}{GL(4, R)} \to GL(4, \mathbb{C}) \tag{48}$$

and

$$\sigma^{\mu\tilde{A}}_{\nu\tilde{B}} = \partial\Sigma^{\tilde{A}}_{\tilde{B}}/\partial A^\nu_\mu\big|_{A^\nu_\mu = \delta^\nu_\mu} \tag{49}$$

[see Kalinowski (1986) for more details]. It is easy to see that

$$\sigma^\mu_\lambda = \tfrac{1}{8}[\gamma^\mu, \gamma_\lambda] \tag{50}$$

satisfies all properties (40)–(42). In Kalinowski (1986) we proved that the Lagrangian (36) or (38) has $U(1)_F$-gauge invariance, which is connected to the compactification of the dilatation subgroup R_+ of $GL_\downarrow(4, R) = R_+ \otimes SL(4, R)$, where

$$\begin{aligned} GL_\downarrow(4, R) &= \{A \in GL(4, R), \det A > 0\} \\ SL(4, R) &= \{A \in GL(4, R), \det A = 1\} \end{aligned} \tag{51}$$

$R_+ = \{e^\rho, \rho \in E\}$, where $\rho = \ln(\det A)$ and the R_+-local gauge group acts in the following way on ψ, $\bar{\psi}$, and $\bar{W}$:

$$\begin{aligned} \bar{W} &\to \bar{W}' = \bar{W} + d\phi \\ \psi &\to \psi' = \exp\left[i\frac{n\varepsilon_F}{8\hbar c}\left(\frac{a}{l_{\text{pl}}}\right)^2 \ln(\det A)\right]\psi \\ \bar{\psi} &\to \bar{\psi}' = \exp\left[-i\frac{n\varepsilon_F}{8\hbar c}\left(\frac{a}{l_{\text{pl}}}\right)^2 \ln(\det A)\right]\bar{\psi} \end{aligned} \tag{52}$$

5. DIRAC EQUATION IN THE NONSYMMETRIC KALUZA-KLEIN THEORY AND IN THE NONSYMMETRIC JORDAN-THIRY THEORY

In this section we deal with a generalization of the Dirac equation on the manifold **P**. We introduce several kinds of derivatives and use them to get a generalization of the Dirac equation.

Let Γ^A, $A=1,2,\ldots,n+4$, be a representation of Clifford algebra for $SO(1,n+3)$ acting in the space representation of D^F, i.e., $\Gamma^A \in C(1,n+3)$ (see Appendix)

$$\{\Gamma^A,\Gamma^B\}=2\bar{g}^{AB}, \qquad \Gamma^A \in \mathscr{L}(\mathbb{C}^K)$$
$$K=4\cdot 2^{[n/2]}, \qquad [n/2]=l \tag{53}$$

where

$$\bar{g}^{AB}=\operatorname{diag}(-1,-1,-1,1,\underbrace{-1,\ldots -1}_{n\times})$$

Let B be a matrix such that

$$\Gamma^{\alpha^+}=B\Gamma^\alpha B^{-1}, \qquad B\in\mathscr{L}(\mathbb{C}^K) \tag{54}$$

The superscript plus sign denotes Hermitian conjugation and we let

$$\bar{\Psi}=\Psi^+B \tag{55}$$

We perform an infinitesimal change of frame θ^A

$$\theta'^A=\theta^A+\delta\theta^A=\theta^A-\varepsilon^A_B\theta^B \tag{56}$$

If the spinor field Ψ corresponds to θ^A and Ψ' to $\theta^{A'}$, then we have

$$\Psi'=\Psi+\delta\Psi=\Psi-\varepsilon^A_B\hat{\sigma}^B_A\Psi$$
$$\Psi'=\bar{\Psi}+\delta\bar{\Psi}=\bar{\Psi}+\Psi\hat{\sigma}^B_A\varepsilon^A_B \tag{57}$$

where

$$\hat{\sigma}^{B\tilde{A}}_{A\tilde{B}}=\partial\hat{\Sigma}^{\tilde{A}}_{\tilde{B}}/\partial A^A_B|_{A^A_B=\delta^A_B} \tag{58}$$

$$\hat{\Sigma}:\quad GL(n+4,R)\to GL(4\cdot 2^{[n/2]},\mathbb{C}) \tag{59}$$

is a homomorphism of Lie groups and $\hat{\sigma}^B_A$ is the differential of $\hat{\Sigma}$ at the unit element. $\hat{\sigma}^B_A$ satisfies the following properties, similar to σ^β_α [see (40)-(42)]:

$$\hat{\sigma}^A_A=0 \tag{60}$$

$$2[\hat{\sigma}^B_A,\hat{\sigma}^D_C]=\delta^D_A\hat{\sigma}^B_C-\bar{g}_{AC}\hat{\sigma}^{BD}+\delta^B_C\hat{\sigma}^D_A-\bar{g}^{BD}\hat{\sigma}_{AC} \tag{61}$$

$$[\hat{\sigma}^B_A,\Gamma^C]=1/2[\delta^C_A\Gamma^B-\bar{g}^{CB}\Gamma_A] \tag{62}$$

where

$$\bar{g}_{AB}\bar{g}^{BC} = \delta^C_A, \qquad \hat{\sigma}^{AB} = \bar{g}^{AC}\hat{\sigma}^B_C$$
$$\Gamma^A = \bar{g}^{AB}\Gamma_B, \qquad \hat{\sigma}_{AB} = \bar{g}_{CB}\hat{\sigma}^C_A \tag{63}$$

We can use the following representation of $\hat{\sigma}^B_A$:

$$\hat{\sigma}^B_A = \tfrac{1}{8}[\Gamma_A, \Gamma^B] \tag{64}$$

Let us consider a covariant derivation of spinor fields Ψ and $\bar{\Psi}$ on **P**:

$$D\Psi = d\Psi + \omega^A_B\hat{\sigma}^B_A\Psi, \qquad D\bar{\Psi} = d\bar{\Psi} - \bar{\Psi}\hat{\sigma}^B_A\omega^A_B \tag{65}$$

with respect to the linear connection ω^A_B from the nonsymmetric Kaluza-Klein theory [Eq. (19)] or from the nonsymmetric Jordan-Thiry theory [see Eq. (26)]. Now we introduce derivative D, i.e., "gauge" derivatives of a new kind [as in Kalinowski (1981a, b, 1982, 1983d, 1984b, 1987). These derivatives may be treated as a generalization of minimal coupling between spinor and gauge (Yang-Mills) fields on **P**:

$$D\Psi = \text{hor}\, D\Psi, \qquad D\bar{\Psi} = \text{hor}\, D\bar{\Psi} \tag{66}$$

(horizontality is understood in the sense of the connection ω on **P**). Using (19) and (26), one gets

$$D\Psi = \bar{D}\Psi + \tfrac{1}{8}\lambda[L^a_{\beta\gamma}\Gamma_a\Gamma^\beta + l_{bd}g^{\alpha\beta}(2H^d_{\gamma\beta} - L^d_{\gamma\beta})\Gamma_\alpha\Gamma^b]\Psi\theta^\gamma$$
$$D\bar{\Psi} = D\bar{\Psi} - \tfrac{1}{8}\lambda\bar{\Psi}[L^a_{\beta\gamma}\Gamma_a\Gamma^\beta + l_{bd}g^{\alpha\beta}(2H^d_{\gamma\beta} - L^d_{\gamma\beta})\Gamma_\alpha\Gamma^b]\theta^\gamma \tag{67}$$

in the nonsymmetric Kaluza-Klein theory and

$$D\Psi = \bar{D}\Psi + \tfrac{1}{8}\rho^2\lambda[L^\alpha_{\beta\gamma}\Gamma_a\Gamma^\beta + l_{bd}g^{\alpha\beta}(2H^d_{\gamma\beta} - L^d_{\gamma\beta})\Gamma_\alpha\Gamma^b]\Psi\theta^\gamma$$
$$D\bar{\Psi} = D\bar{\Psi} - \tfrac{1}{8}\lambda\rho^2\bar{\Psi}[L^a_{\beta\gamma}\Gamma_a\Gamma^\beta + l_{bd}g^{\alpha\beta}(2H^d_{\gamma\beta} - L^d_{\gamma\beta})\Gamma_\alpha\Gamma^b]\theta^\gamma \tag{68}$$

in the nonsymmetric Jordan-Thiry theory, where

$$D\Psi = \text{hor}\, \bar{D}\Psi, \qquad \bar{D}\bar{\Psi} = \text{hor}\, \bar{D}\bar{\Psi}$$

The derivative $\bar{D}$ is a covariant derivative with respect to both $\bar{\omega}^\alpha_\beta$ and "gauge" at once. It introduces an interaction between Yang-Mills and gravitational fields with spinor, in the classical way already known in general relativity or in Einstein-Cartan theory (Trautman, 1973). Now let us turn to the Lagrangian (38) and lift it on the manifold **P**. In order to do this, we have to pass from $\bar{D}$ to D, from ψ, $\bar{\psi}$ to Ψ, $\bar{\Psi}$, and from γ^α to Γ^A. In such a case the Dirac Lagrangian takes the form

$$\mathscr{L} = \frac{i\hbar c}{2}\left[\bar{\Psi} l \wedge \left(D\Psi - \frac{in\varepsilon_F}{3}\bar{W}\Psi\right) + \left(D\bar{\Psi} + \frac{in\varepsilon_F}{3}W\bar{\Psi}\right) \wedge l\Psi\right]$$
$$+ mc\eta\bar{\Psi}\Psi \tag{69}$$

where $l=\Gamma^{\mu}\eta_{\mu}$. After some algebra one gets

$$\mathscr{L}=\mathbf{L}(\Psi, W, \omega)+\frac{il_{\rm pl}}{\alpha_s}q\Big[L^a_{\beta\gamma}\bar{\Psi}\hat{\sigma}^{\beta\gamma}\Gamma_a\Psi-l_{bd}g^{\alpha\beta}[(2H^d_{\gamma\beta}-L^d_{\gamma\beta})\bar{\Psi}\hat{\sigma}^{\gamma}_{.\alpha})^b\Psi]\eta$$

$$-i\frac{l_{\rm pl}}{8\alpha_s}ql_{bd}g^{[\alpha\beta]}H^d_{\alpha\beta}\bar{\Psi}\Gamma^b\Psi\eta \tag{70}$$

where $l_{\rm pl}$ is Planck's length, q is the elementary charge, α_s is a dimensionless coupling constant of the Yang-Mills field, and

$$\mathbf{L}(\Psi, W, \omega)=\frac{i\hbar c}{2}\left[\bar{\Psi}l\wedge\left(\bar{D}\Psi-\frac{in\varepsilon_{\rm F}}{3}\bar{W}\Psi\right)\right]$$

$$+\left[\left(\bar{D}\bar{\Psi}+\frac{in\varepsilon_{\rm F}}{3}\bar{W}\bar{\Psi}\right)\wedge l\Psi\right]+mc\bar{\Psi}\Psi\eta \tag{71}$$

$\mathbf{L}(\Psi, W, \omega)$ describes the interaction between the spinor field and geometry in the nonsymmetric theory of gravitation and Yang-Mills field as in Kalinowski (1986, 1987) for the electromagnetic field.

In the case of the nonsymmetric Jordan–Thiry theory we get

$$\mathscr{L}=\mathbf{L}(\Psi, W, \omega)$$

$$+\frac{il_{\rm pl}}{\alpha_s}\rho^2q[L^d_{\beta\gamma}\bar{\Psi}\hat{\sigma}^{\beta\gamma}\Gamma_a\Psi-l_{bd}g^{\alpha\beta}(2H^d_{\gamma\beta}-L^d_{\gamma\beta})\bar{\Psi}\hat{\sigma}^{\gamma}_{.\alpha}\Gamma^b\Psi]\eta$$

$$-\frac{il_{\rm pl}}{8\alpha_s}\rho^2ql_{bd}g^{[\alpha\beta]}H^d_{\alpha\beta}\bar{\Psi}\Gamma^b\Psi\eta \tag{72}$$

So we see that we get additional terms. They are

$$\frac{il_{\rm pl}}{\alpha_s}q[L^a_{\beta\gamma}\bar{\Psi}\hat{\sigma}^{\beta\gamma}\Gamma_a\Psi-l_{bd}g^{\alpha\beta}(2H^d_{\gamma\beta}-L^d_{\gamma\beta})\bar{\Psi}\hat{\sigma}^{\gamma}_{.\alpha}\Gamma^b\Psi] \tag{73}$$

and

$$-\frac{l_{\rm pl}}{8\alpha_s}gl_{bd}g^{[\alpha\beta]}H^d_{\alpha\beta}\bar{\Psi}\Gamma^b\Psi \tag{74}$$

If one performs the dimensional reduction (35) for $L(\Psi, W, \omega)$, one easily gets (see Appendix)

$$\mathbf{L}(\Psi, W, \omega)\xrightarrow[\text{reduction}]{\text{dimensional}}\sum_{i=1}^{2[n/2]}\mathbf{L}(\psi_i, W, A^e) \tag{75}$$

Thus, one obtains the interaction between spinor fields ψ_i, $i=1,2,\ldots,2^{[n/2]}$, and gravitation and the Yang-Mills field in the already known classical way. It is worth noticing that all fermions ψ_i have the same mass m. Now we turn to new terms (73) and (74). In Kalinowski (1981a, 1987) one deals with the five-dimensional (electromagnetic) case and one interprets the first new term as an interaction of electromagnetic field with a dipole electric moment of the fermion of value $(l_{\rm pl}/\sqrt{\alpha})q$. Now we deal with Yang-Mills fields and should work with a useful concrete representation of Γ^A. We will consider the cases $n=2l$ and $n=2l+1$ separately. If we suppose that the group G is a gauge group which unifies electromagnetic, weak, and strong interactions, then G has a subgroup $U(1)_{\rm el}$ corresponding to electromagnetic interactions after breaking the symmetry. Let dim $G=2l+1$ and let a parameter of the electromagnetic subgroup $U(1)_{\rm el}$ correspond to $A=n+4=2l+5$. Then we turn to the first additional term (73) and perform the dimensional reduction for $d=n+4=2l+5$ and $b=a=n+4=2l+5$. We get

$$\frac{il_{\rm pl}}{\alpha_s} q \sum_{i=1=l}^{2[n/2]} [L^{2l+5}_{\beta\gamma}\bar{\psi}_i\sigma^{\beta\gamma}\gamma_5\psi_i - g^{\alpha\beta}(2H^{2l+5}_{\gamma\beta} - L^{2l+5}_{\gamma\beta})\bar{\psi}_i\sigma^{\gamma}_{.\alpha}\gamma^5\psi_i] \tag{76}$$

where $H^{2l+5}_{\beta\gamma}=F_{\beta\gamma}$ (electromagnetic field) and $L^{2l+5}_{\beta\gamma}=H_{\beta\gamma}$ is the second tensor of the strength of the electromagnetic field (Kalinowski, 1983a, b).

Thus, we get for all fermions a dipole electric moment of order 10^{-31} cm (Kalinowski, 1987). If dim $G=2l$, then this term is forbidden and we have no dipole electric moment of the fermion.

Let us pass to the second additional term (74) and perform the dimensional reduction procedure for $d=b=2l+5$. We get

$$\frac{l_{\rm pl}}{8\alpha_s} qg^{[\alpha\beta]}H^{2l+5}_{\alpha\beta} \sum_{i=1}^{2[n/2]} \bar{\psi}_i\gamma^5\psi_i \tag{77}$$

i.e., we get a pseudomass-like term for every fermion ψ_i (Kalinowski, 1987).

It is possible, as in Kalinowski (1984b), to introduce discrete transformations on **P**, i.e., space reflection Π, time reversal T, charge reflection C, and combined transformations ΠC, $\theta=\Pi CT$. In the case of the Jordan-Thiry theory the only difference will be a factor of ρ^2 in formulas (73), (74), (76), and (77).

APPENDIX

In this appendix we deal with Clifford algebra $C(1, n+3)$ (Atiyah et al., 1964; Cartan, 1966). Due to decomposition rules for $C(1, n+3)$, we write down a useful representation for Γ^A in terms of γ_μ. It is well known that any Clifford algebra can be decomposed into a tensor product of the four elementary Clifford algebras (Atiyah et al., 1964; Cartan, 1966):

$$\begin{aligned} &C(0,1)=\mathbb{C} \text{ (complex numbers)}\\ &C(1,0)=R\oplus R\\ &C(0,2)=H=\text{quaternions}\end{aligned} \tag{A1}$$

We have

$$C(1, n+3) = C(0, 2) \otimes C(1, n+1) \tag{A2}$$

Because we deal with dimensional reduction to space-time E, we define the Clifford algebra $C(1, 3)$ and we easily get

$$\begin{aligned} C(1, n+3) &= \left(\prod_{i=1}^{[n/2]} \otimes C(0, 2) \right) \otimes C(1, 3) \\ &= \left(\prod_{i=1}^{[n/2]} \otimes H \right) \otimes C(1, 3) \end{aligned} \tag{A3}$$

It is well known that either

$$C(1, n+3) = C(1, n+4) \qquad (\text{iff } n+3 = 2l, \quad l \in N_1^\infty) \tag{A4}$$

or

$$C(1, n+2) \simeq C(1, n+3) \qquad (\text{iff } n+3 = 2l+1, \quad l \in N_1^\infty)$$

Let $\gamma_\mu \in \mathscr{L}(\mathbb{C}^4)$, $\mu = 1, 2, 3, 4$, be Dirac matrices obeying conventional relations

$$(\gamma_\mu, \gamma_\nu) = 2\eta_{\mu\nu} \tag{A5}$$

$$\begin{aligned} &\eta_{\mu\nu} = \operatorname{diag}(-1, -1, -1, +1) \\ &\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4, \qquad \gamma_5^2 = -1 \end{aligned} \tag{A6}$$

and let $\sigma_i \in \mathscr{L}(\mathbb{C}^2)$, $i = 1, 2, 3$, be Pauli matrices obeying conventional relations as well:

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij} \tag{A7}$$

$$[\sigma_i, \sigma_j] = \in_{ijk}\sigma_k \tag{A8}$$

We introduce also the following notations: $\mathbb{1} \in \mathscr{L}(\mathbb{C}^2)$ is a 2×2 unit matrix and $\mathscr{I} \in \mathscr{L}(\mathbb{C}^4)$ is a 4×4 unit matrix. Thus, one performs on the decomposition (A3) and easily gets

$$\Gamma^\mu = \eta^\mu \otimes \left(\prod_{i=1}^{[n/2]} \otimes \sigma_{\mathbb{1}} \right) \tag{A9}$$

or

$$\Gamma^\mu = \begin{pmatrix} 0 & & \cdot\,\gamma^\mu \\ & \therefore & \\ \gamma^\mu\,\cdot & & 0 \end{pmatrix} \tag{A10}$$

For $A \neq \mu$ one gets (in the case $n=2l$)

$$\Gamma^{2p+1} = i\mathscr{J}\otimes\left(\prod_{i=1}^{p-2}\otimes\mathbb{1}\right)\otimes\tilde{\sigma}_3\otimes\left(\prod_{i=1}^{l-p+1}\otimes\sigma_1\right)$$

$$\Gamma^{2p+2} = i\mathscr{J}\otimes\left(\prod_{i=1}^{p-2}\otimes\mathbb{1}\right)\otimes\sigma_2\otimes\left(\prod_{i=1}^{l-p+1}\otimes\sigma_i\right) \tag{A11}$$

where $4<2p+1<2p+2\leq n+4=2l+2$.

In the case $n=2l$ we define also the matrix

$$\Gamma^{n+5} = ii^{3(l+1)}\prod_{A=1}^{n+4}\Gamma^A = (\gamma^5)\otimes\left(\prod_{i=1}^{l}\otimes\sigma_1\right) = \Gamma^{2l+5} \tag{A12}$$

or

$$\Gamma^{n+5} = \begin{pmatrix} 0 & & \gamma^5 \\ & \ddots\!\!\!\!\!\!\!\!\!\;\;\cdot^{\cdot^{\cdot}} & \\ \gamma^5 & & 0 \end{pmatrix} \tag{A13}$$

where $n=2l$, $l\in N_1^\infty$.

If $n=2l+1$, we have $\tilde{\Gamma}^A = \Gamma^A$, $A=1,2,\ldots,2l+4$:

$$\tilde{\Gamma}^{n+4} = \Gamma^{2l+5} = \begin{pmatrix} 0 & & \gamma^5 \\ & \ddots\!\!\!\!\!\!\!\!\!\;\;\cdot^{\cdot^{\cdot}} & \\ \gamma^5 0 & & 0 \end{pmatrix} \tag{A14}$$

It is easy to check that

$$(\Gamma^{2l+5})^2 = -1, \qquad \{\tilde{\Gamma}^A, \Gamma^{2l+5}\} = 0 \qquad \text{for} \quad A\neq 2l+5 \tag{A15}$$

$$B = \bar{B}\otimes\left(\prod_{i=1}^{[n/2]}\otimes\sigma_1\right), \qquad \gamma^{\mu+} = \bar{B}\gamma^\mu B^{-1} \tag{A16}$$

ACKNOWLEDGMENTS

I would like to thank Prof. J. W. Moffat and Drs. G. Kunstatter and R. B. Mann for their kind hospitality during my stay at the Physics Department of the University of Toronto.

REFERENCES

Atiyah, M. F., Bott, R., and Shapiro, A. (1984). Clifford modules, *Topology Supplement*, **1**, 3 (1964).

Barut, O., and Raczka, R. (1977). *Theory of Group Representations and Applications*, PWN, Warsaw.

Cartan, E. (1966). *The Theory of Spinors*, Hermann, Paris.

Cho, Y. (1975). Higher dimensional unification of gravitation and gauge theories, *Journal of Mathematical Physics*, **16**, 2029.

Furlan, P., and Raczka, R. (1980). A new approach to unified field theories, JCTP preprint JC/80/181, Trieste.

Kalinowski, M. W. (1981a). *PC*-nonconservation and a dipole electric moment for fermion in the Klein-Kaluza theory, *Acta Physica Austriaca*, **53**, 229.

Kalinowski, M. W. (1981b). Gauge fields with torsion, *International Journal of Theoretical Physics*, **20**, 563.

Kalinowski, M. W. (1982). Rarita Schwinger field in the nonabelian Kaluza-Klein theories, *Journal of Physics A*, **15**, 2441.

Kalinowski, M. W. (1983a). The nonsymmetric Kaluza-Klein theory, *Journal of Mathematical Physics*, **24**, 1835.

Kalinowski, M. W. (1983b). The nonsymmetric-nonabelian Kaluza-Klein theory, *Journal of Physics A*, **16**, 1669.

Kalinowski, M. W. (1983c). The nonsymmetric Jordan-Thiry theory, *Canadian Journal of Physics*, **61**, 844.

Kalinowski, M. W. (1983d). $\frac{3}{2}$-Spinor field in the Klein-Kaluza theory, *Acta Physica Austriaca*, **55**, 167.

Kalinowski, M. W. (1983e). Vanishing of the cosmological constant in nonabelian Kaluza-Klein theories, *International Journal of Theoretical Physics*, **22**, 385.

Kalinowski, M. W. (1984a). The nonsymmetric-nonabelian Jordan-Thiry theory, *Il Nuovo Cimento*, **LXXXA**, 47.

Kalinowski, M. W. (1984b). Spinor fields in nonabelian Kaluza-Klein theories, *International Journal of Theoretical Physics*, **23**, 131.

Kalinowski, M. W. (1986). The minimal coupling scheme for Dirac's field in the nonsymmetric theory of gravitation, *International Journal of Modern Physics*, **A1**, 227.

Kalinowski, M. W. (1987). *CP*-Nonconservation and electric dipole moment of fermion in nonsymmetric Kaluza-Klein theory, *International Journal of Theoretical Physics*, **26**, 21.

Kaluza, Th. (1921). *Sitzgsberichte Preussische Akademie der Wissenschaften*, **1921**, 966.

Kerner, R. (1968). Generalization of Kaluza-Klein theory for an arbitrary nonabelian gauge group, *Annales de l'Institut Henri Poincaré A*, **IX:**143 (1968).

Kim, C. W. (1987). Grand unification. A review and implications in cosmology, preprint HV-HET 8006.

Kobayashi, S., and Nomizu, K. (1963). *Foundations of Differential Geometry*, Vols. I and II, Wiley, New York.

Lichnerowicz, A. (1955a). *Théorie relativistes de la gravitation et de l'électromagnetisme*, Masson, Paris.

Lichnerowicz, A. (1955b). *Théorie globale des connexions et de group d'holomie*, Cremonese, Rome.

Moffat, J. W. (1979). New theory of gravitation, *Physical Review D*, **19**, 3557.

Moffat, J. W. (1981). Gauge invariance and string interactions in a generalized theory of gravitation, *Physical Review D*, **23**, 2870.

Moffat, J. W. (1982). Generalized theory of gravitation and its physical consequences, in *Proceedings of the VII International School of Gravitation and Cosmology, Erice, Sicily*, V. de Sabatta, ed., World Scientific, Singapore.

Pati, J. C. (1980). Talk presented at the Symposium on Grand Unified Theories, JCTP, Trieste.

Rayski, J. (1977). Unitary spin, colour and unified theories, *Acta Physica Austriaca Supplementum*, **XVIII**, 963.

Rayski, J. (1965). Unified theory and modern physics, *Acta Physica Polonica*, **XXVIII**, 89.

Thirring, W. (1972). Five-dimensional theories and *CP*-violation, *Acta Physica Austriaca Supplementum*, **XVIII**, 463.

Trautman, A. (1970). Fibre bundles associated with space-time, *Reports of Mathematical Physics*, **1**, 29.

Trautman, A. (1973). On the structure of Einstein-Cartan equations, *Symposia Mathematica*, **12**, 139 (1973).